RESEARCH ARTICLE

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Lyapunov Spectrum Characterization of Irregular Oscillations in Climate-Coupled Dynamic Systems

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Abstract

Climate-coupled dynamic systems produce irregular oscillations when fast atmospheric variability interacts nonlinearly with slower oceanic and feedback-driven processes. Such irregularity is important because waveform appearance alone does not show whether the system remains weakly predictable or has entered an unstable regime. Nonlinear climate studies increasingly use Lyapunov-based methods to diagnose instability and predictability loss, but the full Lyapunov spectrum is still used less often than leading-error indicators or event-specific measures. This creates a gap because the spectrum can distinguish bounded modulation, weak irregularity, and chaotic variability more clearly than ordinary time-series analysis. This article develops a reduced slow-fast climate-coupled nonlinear model and characterizes its oscillatory regimes through Lyapunov-spectrum analysis and trajectory-based interpretation. The results show that increasing coupling strength and slow-memory feedback move the system from regular oscillation through a weakly irregular regime and then into chaotic variability marked by a positive leading Lyapunov exponent. Overall, the study shows that irregular climate oscillations are best understood as spectral transitions in stability structure.

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1. Introduction

Climate-coupled dynamic systems arise from interacting atmosphere, ocean, land, cryosphere, and radiative feedback processes whose characteristic timescales are strongly separated and whose mutual influence is nonlinear. This multiscale coupling produces structurally rich variability [1], and nonlinear analyses of internal climate variability have shown that dynamical descriptions often resolve regime structure more effectively than purely statistical summaries [2]. Because of this, climate oscillations frequently appear irregular even when the underlying system remains low dimensional in its effective dynamics. Irregular variability in such systems is therefore best interpreted as a consequence of coupled nonlinear evolution rather than as a simple fluctuation around a mean state.

Predictability in coupled climate systems depends not only on the size of initial perturbations but also on how the evolving trajectory is embedded within the attractor geometry of the system. Ocean-atmosphere predictability studies have shown that attractor-radius concepts can reveal limitations that remain hidden in standard trajectory diagnostics [3]. Multi-model atmosphere-ocean analyses have also shown that coupling modifies stability structure and therefore changes the practical prediction horizon of the evolving state [4]. These findings imply that irregular oscillations and predictability loss are dynamically inseparable. A framework that measures trajectory instability directly is therefore more appropriate than one based only on waveform description.

Lyapunov-based methods are especially attractive in this setting because they quantify instability in the tangent dynamics rather than only in the observable signal. Nonlinear local Lyapunov theory has been used successfully to interpret decadal variations in ENSO predictability [5]. Finite-time studies of chaotically forced ocean-atmosphere systems have likewise shown that crisis-like transitions correspond to changing local instability structure rather than to simple amplitude growth [6]. This means that irregular climate oscillations should not be identified only by visual irregularity or spectral broadening. They should instead be interpreted through the stability spectrum that governs the growth or decay of perturbations along the evolving trajectory.

A clear gap remains because many existing studies focus on leading predictability indicators or event-specific instability measures rather than on the Lyapunov spectrum as an organized description of regime structure. The spectrum reveals how many dynamically active directions are present, how close the system is to marginal instability, and whether irregular oscillation corresponds to weak modulation or fully developed chaos. This article addresses that gap by developing a reduced climate-coupled nonlinear model and characterizing its oscillatory regimes through Lyapunov-spectrum analysis. The specific contribution is the explicit linking of regular, weakly irregular, and chaotic climate-coupled oscillations to measurable changes in spectral stability structure under slow-fast feedback interaction. In this way, the study provides a more rigorous route for interpreting climate irregularity, predictability loss, and coupling-driven regime transitions.

2. Methodology

The model is formulated as a reduced climate-coupled nonlinear system representing fast atmospheric variability, slower ocean-memory adjustment, and an internal feedback process that modulates energy exchange. The state variable $x(t)$is interpreted as a fast atmospheric anomaly, $y(t)$as a slower oceanic memory or storage state and $z(t)$as a feedback variable representing radiative-thermodynamic modulation. Low-order nonlinear systems of this kind remain useful in coupled climate predictability analysis because they preserve the essential instability structure of atmosphere-ocean interaction [7]. Decadal predictability studies using nonlinear Lyapunov-based frameworks also support the use of reduced slow-fast formulations when the goal is to isolate the dynamical mechanisms that degrade predictability [8]. The aim here is therefore not full climate realism, but a controlled nonlinear setting in which irregular oscillation can be tied directly to spectral instability.

The governing equations are written as

$$\frac{dx}{dt} = a(y - x) - xz + \sigma x^3, \qquad (1)$$

$$\frac{dy}{dt} = bx - y - \mu yz, \qquad (2)$$

$$\frac{dz}{dt} = \varepsilon(cxy - dz + F), \qquad (3)$$

where a, b, c and $d$are coupling coefficients, σcontrols nonlinear atmospheric self-interaction, μcontrols state-dependent damping through feedback coupling, $F$is an imposed forcing term, and $\varepsilon \ll 1$sets the slow timescale of the feedback variable. The pair (x, y)forms the fast-to-intermediate climate subsystem, while $z$provides slow modulation of the oscillatory state. Local predictability studies of extreme climatic events have shown that instability can be strongly state dependent even when the governing regime appears statistically stationary [9]. Hyperbolicity-oriented analyses of persistent atmospheric regimes also support the need to interpret variability through evolving stability structure rather than through amplitude alone [10]. This slow-fast arrangement is what allows the model to generate amplitude modulation, delayed feedback, and transitions from regular to irregular oscillation.

Numerical integration is performed over a long horizon so that transient behavior is removed and only asymptotic oscillatory regimes are analyzed. The principal control parameters are the atmosphere-ocean coupling strength a, the nonlinear self-interaction coefficient σ and the slow-memory parameter ε. Practical predictability analyses of sudden stratospheric warming have shown that nonlinear growth rates can change substantially across nearby dynamical states [11]. Nonlinear studies of seasonal predictability barriers in sea-surface-temperature anomalies have likewise demonstrated that irregularity can intensify through state-dependent instability rather than through simple mean-state change [12]. The parameter sweep is therefore designed to capture the transition from regular oscillation to weak irregularity and then to chaos under progressively stronger coupling and feedback interaction. This makes the regime structure of the model directly interpretable in dynamical rather than purely visual terms.

The stability of the evolving trajectory is characterized by integrating the tangent linear system simultaneously with the nonlinear state. Writing the state vector as $\mathbf{X} = [x, y, z]^T$, the tangent perturbation $\delta\mathbf{X}$evolves according to

$$\frac{d}{dt}\delta\mathbf{X} = J(\mathbf{X})\,\delta\mathbf{X}, \qquad (4)$$

where the Jacobian matrix is

$$J(\mathbf{X}) = \begin{bmatrix} -a - z + 3\sigma x^2 & a & -x \\ b & -1 - \mu z & -\mu y \\ \varepsilon c y & \varepsilon c x & -\varepsilon d \end{bmatrix}. \qquad (5)$$

Repeated orthonormalization of tangent vectors yields the ordered Lyapunov spectrum

$$\lambda_1 \geq \lambda_2 \geq \lambda_3. \qquad (6)$$

A non-positive leading exponent indicates asymptotic stability or near-neutral oscillation, while a positive leading exponent indicates instability-driven sensitivity to initial conditions. This spectrum therefore provides the formal basis for distinguishing regular, weakly irregular, and chaotic climate-coupled regimes.

To relate spectral instability to observed waveform behavior, the study computes a finite-time irregularity measure from the fast climatic state,

$$\mathcal{I} = \frac{1}{T}\int_0^T |\, x(t) - \bar{x} \,|\; dt, \qquad (7)$$

where $\bar{x}$is the long-time average of $x(t)$. Connectivity-oriented analyses of Arctic weather variability support the use of state-sensitive descriptors when large-scale irregularity is distributed across interacting climate components [13]. The quantity $\mathcal{I}$is not used as a replacement for the Lyapunov spectrum, but as a companion descriptor that helps separate visibly modulated oscillation from genuinely instability-driven irregularity. Regime classification is then performed by combining trajectory morphology with the sign pattern of the spectrum. This allows the results section to distinguish periodic, weakly irregular, and chaotic regimes through a common dynamical framework.

3. Results and Discussion

The simulated climate trajectories do not lose regularity abruptly. Instead, the coupled system first enters a regime of slowly modulated oscillation in which the dominant atmospheric anomaly retains a recognizable cycle, but the envelope and phase begin to drift. Figure 1 shows that the weakest coupling case remains close to a repeatable oscillatory orbit, consistent with a dynamically restrained atmosphere-ocean exchange. As the coupling strength increases, the oscillation becomes uneven in both spacing and amplitude, and under the strongest interaction level the trajectory no longer revisits a stable repeating path. The important point is that this irregularity emerges progressively from internal climate coupling rather than from an imposed random disturbance.

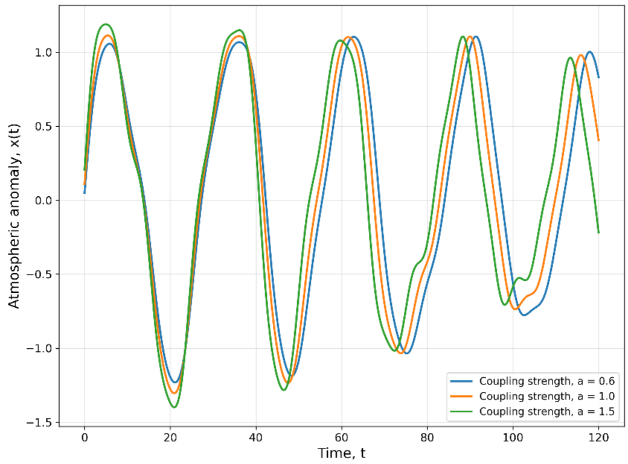

Fig. 1. Irregular oscillatory trajectories under varying climate-coupling strength

The Lyapunov spectrum gives that transition a precise dynamical meaning. In the regular regime, the leading exponent remains non-positive, which indicates that perturbations do not amplify in a sustained way and that the coupled trajectory remains locally attracting. In the intermediate regime, the leading exponent approaches zero while the second exponent becomes less negative, implying a weakened restoring structure and an increased sensitivity of the evolving climatic state. Figure 2 shows that this near-marginal configuration corresponds to the weakly irregular oscillations identified in the trajectory analysis. Under stronger coupling, the leading exponent becomes positive and remains clearly separated from the stable and marginal cases, marking the onset of instability-driven unpredictability.

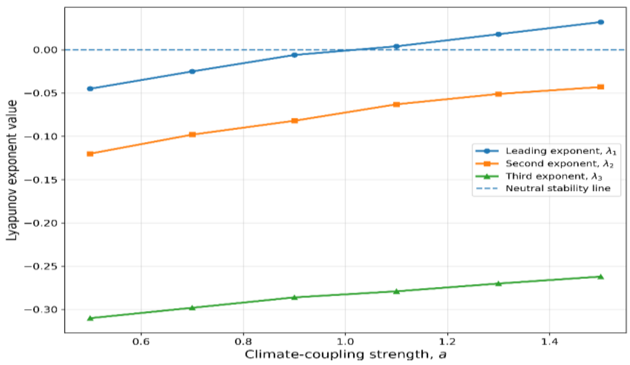

Fig. 2. Lyapunov-spectrum variation across climate-coupled regimes

Slow feedback plays a decisive role in producing this spectral transition. The slow variable does not simply delay the fast oscillation; it repeatedly reshapes the local phase-space geometry through alternating intervals of damping and amplification. This mechanism explains the broadened oscillatory excursions observed in the strongly coupled cases and the simultaneous shift of the leading Lyapunov exponent into the positive range. In climate terms, reduced predictability in the model is not caused by larger oscillation amplitude alone, but by an increasingly unstable exchange between fast atmospheric adjustment and slower memory-driven modulation. The source of irregularity is therefore embedded in the feedback architecture of the coupled system itself.

4. Conclusion

Irregular oscillations in coupled climate systems do not arise here as random departures from an otherwise regular cycle. They emerge through a progressive destabilization of the coupled atmosphere-ocean state as slow feedback begins to modulate the fast oscillatory component more strongly. The analysis shows that the most meaningful change is not simply the visual loss of periodicity, but the shift in the Lyapunov spectrum from a contracting structure to one that admits sustained perturbation growth. The weakly irregular regime is particularly important in this regard, because it marks the point at which the climate trajectory is no longer dynamically simple even though it has not yet entered a fully chaotic state. This makes the transition zone itself scientifically significant, since it captures the onset of reduced predictability before complete instability is established.

The Lyapunov spectrum acts less as a supplementary diagnostic and more as the organizing framework of the problem. It separates bounded oscillatory variability from genuinely unstable variability and clarifies how slow-memory feedback changes the predictability character of the system. The value of the present study therefore lies in showing that coupled climate irregularity can be interpreted through a hierarchy of dynamical regimes rather than through a binary distinction between regular and chaotic behavior. That viewpoint is especially useful for reduced climate modeling, where the objective is not only to reproduce variability, but to understand when and why predictability deteriorates as feedback structure changes.

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